Proceedings of the Jangjeon Mathematical Society
21 (2018), No. 1. pp. 23 – 31
www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2018.21.1.23

NEW SEPARATION AXIOMS ON CLOSURE SPACES GENERATED BY RELATIONS

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ABSTRACT. In this paper, some higher separation axioms on closure spaces are introduced by using binary relations. Further, characterizations, subspaces and preservation under mapping of the newly defined spaces are also studied.

1. INTRODUCTION AND PRELIMINARIES

It is evident from recent developments that topological structures which are more general than usual topology has many applications in allied branches of Science and Engineering. Čech closure space defined by Čech [3] being a generalized topological space has been utilized in digital topology [4, 10, 11], a theory developed in late 1960s for the study of geometric and topological properties of digital images [5, 6, 8, 9]. Čech closure spaces are obtained from the Kuratowski ones by omitting the conditions of idempotency. In 2006, Allam, Bakeir and Abo-Tabl [1] introduced a new approach to define closure spaces through relations and studied lower separation axioms, continuous functions and subspaces on closure spaces generated by relations. In 2008, the same authors [2] introduced some methods to generate topologies via binary relations. G. Liu [7] in 2010, utilized the notion of aftersets and foresets [1] which is renamed as R-left and R-right relative sets in [7]. In the present paper, we introduced and studied higher separation axioms on closure spaces generated by relations.
Let X be any set then a relation on X is a subset of $X \times X$, i.e., $R \subseteq X \times X$. The formula $(x, y) \in R$ is abbreviated as xRy means that x is in relation R with y.

Definition 1.1. [3] A closure space is a pair (X, cl), where X is any set and $cl : P(X) \to P(X)$ is a mapping called as closure operator associating each subset $A \subseteq X$ with a subset $cl(A) \subseteq X$, such that

(1) $cl(\phi) = \phi$,
(2) $A \subseteq cl(A)$,
(3) $cl(A \cup B) = cl(A) \cup cl(B)$.

The subset $cl(A) \subseteq X$ is called as closure of A.

Definition 1.2. [1] If R is a relation on X, then the aftersets of $x \in X$ is xR where $xR = \{y : xRy\}$ and foresets of $x \in X$ is Rx where $Rx = \{y : yRx\}$.

Remark 1.3. The above aftersets and foresets are represented as $r_R(x)$ and $l_R(x)$ respectively in [7].

2010 *Mathematics Subject Classification.* 04A05, 54A05, 54B05, 54C05, 54D15.
Key words and phrases. aftersets, foresets, binary relation, normal, regular, subspace, continuous function, homeomorphism, closure subspace, T_1 space.

Definition 1.4. [1] Let R be any binary relation on X, a set $\langle p\rangle R$ is the intersection of all aftersets containing p, i.e.,

$$\langle p\rangle R = \begin{cases} \bigcap\limits_{p\in xR} (xR) & \text{if there exists } x \text{ such that } p \in xR \ , \\ \phi & \text{otherwise.} \end{cases}$$

Also, $R\langle p\rangle$ is the intersection of all foresets containing p, i.e.,

$$R\langle p\rangle = \begin{cases} \bigcap\limits_{p\in Rx} (Rx) & \text{if there exists } x \text{ such that } p \in Rx \ , \\ \phi & \text{otherwise.} \end{cases}$$

Definition 1.5. [1] Let X be any set and $R \subseteq X \times X$ be any binary relation on X. The relation R gives rise to a closure operation cl_R on X as follows:

$$cl_R(A) = A \cup \Big\{x \in X : \ \langle x\rangle R \cap A \neq \phi\Big\}$$

The set along with the operator cl_R is a closure space. In the closure space (X, cl_R), a set A is closed [1] if $cl_R(A) = A$.

Lemma 1.6. [1] *In a closure space (X, cl_R) the open sets are precisely the unions $\bigcup\limits_{x\in A}(N_R(x))$ for all $A \subseteq X$ and*

$$N_R(x) = \begin{cases} \langle x\rangle R & \text{if } \langle x\rangle R \neq \phi, \\ \{x\} & \text{if } \langle x\rangle R = \phi. \end{cases}$$

Lemma 1.7. [1] *For any binary relation R on X if $x \in \langle y\rangle R$, then $\langle x\rangle R \subseteq \langle y\rangle R$.*

Theorem 1.8. *In a closure space (X, cl_R), if $\langle x\rangle R$ is open, then $X - \langle x\rangle R$ is closed.*

Proof. We know that $(X - \langle x\rangle R) \subset cl_R(X - \langle x\rangle R)$. We have to prove $cl_R(X - \langle x\rangle R) \subset (X - \langle x\rangle R)$. Suppose $cl_R(X - \langle x\rangle R) \not\subseteq (X - \langle x\rangle R)$ then there exist $y \in cl_R(X - \langle x\rangle R)$ such that $y \notin (X - \langle x\rangle R)$ implies $y \in \langle x\rangle R$. Then by using lemma 1.7 $\langle y\rangle R \subseteq \langle x\rangle R$ which implies $\langle y\rangle R$ does not intersect $(X - \langle x\rangle R)$ implies $y \notin cl_R(X - \langle x\rangle R)$. Which is a contradiction. So $cl_R(X - \langle x\rangle R) \subset (X - \langle x\rangle R)$. Hence $cl_R(X - \langle x\rangle R) = (X - \langle x\rangle R)$. □

Theorem 1.9. *In a closure space (X, cl_R), $cl_R(\langle y\rangle R)$ is the smallest closed set containing $\langle y\rangle R$.*

Proof. Let B be the smallest closed set containing $\langle y\rangle R$. Since B is closed, $cl_R(B) = B$ and $\langle y\rangle R \subset B = cl_R(B)$. If $cl_R(\langle y\rangle R) \not\subseteq cl_R(B)$ then there exists $x \in cl_R(\langle y\rangle R)$ such that $x \notin cl_R(B)$ implies $x \notin B$. But $x \in cl_R(\langle y\rangle R)$ implies $\langle x\rangle R \cap \langle y\rangle R \neq \phi$. Thus $\langle x\rangle R \cap B \neq \phi$ as $\langle y\rangle R \subset B$. Which is a contradiction. So $cl_R(\langle y\rangle R) \subset cl_R(B)$. Hence $cl_R(\langle y\rangle R)$ is the smallest closed set containing $\langle y\rangle R$. □

Definition 1.10. [1] Let R be any binary relation, then a closure space (X, cl_R) is called T_1-space if and only if for every two distinct points $x, y \in X$ both $x \notin \langle y\rangle R$ and $y \notin \langle x\rangle R$ are holds.

Definition 1.11. [1] Let $A \subseteq X$ and $R_A \subseteq R$, then (A, cl_{R_A}) is called a closure subspace of the closure space (X, cl_R) if $\langle x\rangle R_A = \langle x\rangle R \cap A$ for all $x \in A$.

Remark 1.12. [3] Let Y be a subspace of a closure space X. Then
(a) A is closed (open) in X implies $Y \cap A$ is closed (open) in Y.
(b) Y is closed in X and A is closed set in Y implies A is closed in X.

Lemma 1.13. [1] *Let (A, cl_{R_A}) be a closure subspace of a closure space (X, cl_R), then $\langle x\rangle R_A = \langle x\rangle R \cap A$ for all $x \in A$ if and only if $cl_{R_A}(B) = cl_R(B) \cap A$ for all $B \subseteq A$.*

Theorem 1.14. [1] *Let (X, cl_R) be a closure space and $A \subseteq X$ then (A, cl_{R_A}) is a closure subspace if and only if $cl_{R_A}(B) = cl_R(B) \cap A$ for all $B \subseteq A$.*

2. Normality

Definition 2.1. Let R be any binary relation on X, then a closure space (X, cl_R) is said to be normal if for every two disjoint closed sets A and B in X, there exist two disjoint open sets U and V containing A and B respectively.

Definition 2.2. Let R be any binary relation on X, then a closure space (X, cl_R) is said to be strongly normal if for every two disjoint closed sets A and B there exist x, y such that $A \subset (\langle x\rangle R)$, $B \subset (\langle y\rangle R)$ and $\langle x\rangle R \cap \langle y\rangle R = \phi$.

Example 2.3. *Let X be the set of natural numbers and R be the relation defined on X as aRb if and only if $a = b$, for every $a \in X$ and $b = a + n$, for $a \geqslant 2$, $n = 1, 2, 3, \ldots$. Here the aftersets are $1R = \{1\}$ and $iR = \{i + n : n = 0, 1, 2, \ldots\}$, for $i \geq 2$. Thus, $\langle 1\rangle R = \{1\}$ and $\langle i\rangle R = \{i + n : n = 0, 1, 2 \ldots\}$, for $i \geq 2$. In this space only disjoint closed sets are $\{1\}$ and $\{2\}$ which can be separated by $\langle 1\rangle R$ and $\langle 2\rangle R$ respectively. Hence (X, cl_R) is a strongly normal closure space.*

Observation 2.4. *Every strongly normal space is normal.*

Generally, a normal space need not be strongly normal as shown in the following examples.

Example 2.5. *Let $X = \{a, b, c, d\}$ and R be any binary relation on X, where $R = \Big\{(a, a),\ (a, c),\ (a, b),\ (b, a),\ (b, b)\ (b, d), (c, d),\ (c, a),\ (d, d),\ (d, c)\Big\}$. Then $\langle a\rangle R = \{a\}$, $\langle b\rangle R = \{a, b\}$, $\langle c\rangle R = \{c\}$, $\langle d\rangle R = \{d\}$ and $cl_R(\{a\}) = \{a, b\}$, $cl_R(\{b\}) = \{b\}$, $cl_R(\{c\}) = \{c\}$, $cl_R(\{d\}) = \{d\}$, $cl_R(\{a, b\}) = \{a, b\}$, $cl_R(\{a, c\}) = \{a, b, c\}$, $cl_R(\{a, d\}) = \{a, b, d\}$, $cl_R(\{b, c\}) = \{b, c\}$, $cl_R(\{b, d\}) = \{b, d\}$, $cl_R(\{c, d\}) = \{c, d\}$, $cl_R(\{a, b, c\}) = \{a, b, c\}$, $cl_R(\{a, b, d\}) = \{a, b, d\}$, $cl_R(\{a, c, d\}) = X$, $cl_R(\{b, c, d\}) = \{b, c, d\}$, $cl_R(X) = X$, $cl_R(\phi) = \phi$. Here the finite closure space (X, cl_R) is normal because for every two disjoint closed sets there exist disjoint open sets separating the closed sets but not strongly normal because for two closed sets $A = \{a, b\}$ and $B = \{c, d\}$ there doesn't exist disjoint $\langle x\rangle R$ and $\langle y\rangle R$ separating them.*

Example 2.6. *Let X be the set of natural numbers and R be the relation defined on X as aRb if and only if $b = a \pm 1$. Here aftersets are $1R = \{2\}$ and $iR = \{i-1, i+1\}$ for every $i \geqslant 2$. Thus, $\langle 1\rangle R = \{1, 3\}$ and $\langle i\rangle R = \{i\}$ for every $i \geq 2$. In this space, those subset of X which contains 3 but not 1 are not closed sets. But rest of the subsets of X are closed sets. Here, the infinite closure space (X, cl_R) is normal but (X, cl_R) is not strongly normal because for two closed sets $A = \{1\}$ and $B = \{2, 4\}$ there doesn't exist disjoint $\langle x\rangle R$ and $\langle y\rangle R$ separating A and B respectively.*

Theorem 2.7. *A closure space (X, cl_R) is strongly normal if and only if for every closed set A contained in $\langle x\rangle R$, for some x there exists y such that $A \subset \langle y\rangle R \subset cl_R(\langle y\rangle R) \subset \langle x\rangle R$.*

Proof. Let (X, cl_R) be a strongly normal space and let $A = cl_R(A)$ be a closed set contained in $\langle x\rangle R$ for some $x \in X$. So A and $B = (X - \langle x\rangle R)$ are two disjoint closed sets in X. Thus by strong normality of X, there exist disjoint $\langle y\rangle R$ and $\langle w\rangle R$ such that $A \subset \langle y\rangle R, B \subset \langle w\rangle R$. Therefore $A \subset \langle y\rangle R$ and $\langle y\rangle R \subset (X - \langle w\rangle R)$. Since $(X - \langle w\rangle R) \subset (X - B) = \langle x\rangle R$, $A \subset \langle y\rangle R \subset (X - \langle w\rangle R) \subset \langle x\rangle R$. As $(X - \langle w\rangle R)$ is a closed set containing $\langle y\rangle R$ and $cl_R(\langle y\rangle R)$ is the smallest closed set containing $\langle y\rangle R, A \subset \langle y\rangle R \subset cl_R(\langle y\rangle R) \subset (X - \langle w\rangle R) \subset \langle x\rangle R)$ implies $A \subset \langle y\rangle R \subset cl_R(\langle y\rangle R) \subset (\langle x\rangle R)$. Conversely, let $A = cl_R(A)$ and $B = cl_R(B)$ be two closed sets in X. Let $\langle x\rangle R = X - B$ implies $A \subset \langle x\rangle R$. By given hypothesis there exists y such that $A \subset \langle y\rangle R \subset cl_R(\langle y\rangle R) \subset \langle x\rangle R$. Therefore, $\langle y\rangle R$ and $(X - cl_R(\langle y\rangle R))$ are two disjoint open sets containing A and B respectively. Thus X is strongly normal. □

Theorem 2.8. *In a strongly normal space every pair of disjoint closed sets can be separated by disjoint open sets whose closures are disjoint.*

Proof. Let X be a strongly normal space and $cl_R(A) = A$, $cl_R(B) = B$ be two disjoint closed sets in X. Since $cl_R(A) \cap cl_R(B) = \phi$, we have $cl_R(A) \subset (X - cl_R(B))$, where $(X - cl_R(B))$ is open. By strong normality of X, there exists an open set $\langle x\rangle R$ of $cl_R(A)$ such that $cl_R(A) \subset \langle x\rangle R \subset cl_R(\langle x\rangle R) \subset (X - cl_R(B))$. So $cl_R(\langle x\rangle R) \cap cl_R(B) = \phi$. Since $cl_R(B) \subset (X - cl_R(\langle x\rangle R))$ is an open set containing $cl_R(B)$. Again by using strong normality, there exists an open set $\langle y\rangle R$ of $cl_R(B)$ such that $cl_R(B) \subset \langle y\rangle R \subset cl_R(\langle y\rangle R) \subset (X - cl_R(\langle x\rangle R))$ which implies $cl_R(\langle y\rangle R) \cap cl_R(\langle x\rangle R) = \phi$. □

Remark 2.9. The infinite closure space discussed in Example 2.3 is a strongly normal closure space in which only disjoint closed sets $\{1\}$ and $\{2\}$ can be separated by disjoint open sets $\langle 1\rangle R$ and $\langle 2\rangle R$ whose closures are $\{1\}$ and $\{2, 3, 4, ...\}$ respectively.

The following example establishes that closure subspace of a strongly normal space need not be strongly normal.

Example 2.10. *Let $X = \{a, b, c, d\}$ and $R = \Big\{(a, a), (a, c), (b, a), (b, b), (c, a), (d, a), (d, b), (d, c), (d, d)\Big\}$ be a binary relation on X. Let $Y = \{a, b, c\}$ be a subset of X, then $R_Y = \Big\{(a, a), (a, c), (b, a), (b, b), (c, a)\Big\}$ is a subset of R. Here, $\langle a\rangle R = \{a\}$, $\langle b\rangle R = \{a, b\}$, $\langle c\rangle R = \{c, a\}$, $\langle d\rangle R = \{a, b, c, d\}$ and $\langle a\rangle R_Y = \{a\}$, $\langle b\rangle R_Y = \{a, b\}$, $\langle c\rangle R_Y = \{a, c\}$. Note that (X, cl_R) is a closure space which is strongly normal but (Y, cl_{R_Y}) is not strongly normal because for two disjoint closed sets $A = \{b\} = cl_{R_Y}(A)$ and $B = \{c\} = cl_{R_Y}(B)$ there doesn't exist disjoint $\langle x\rangle R_Y$ and $\langle y\rangle R_Y$ separating A and B. Hence subspace of a strongly normal space need not be strongly normal.*

Theorem 2.11. *Closed closure subspace of a strongly normal space is strongly normal.*

Proof. Let Y be a closed closure subspace of X, $A = cl_{R_Y}(A)$ and $B = cl_{R_Y}(B)$ be two disjoint closed sets in Y. Since Y is closed in X and $cl_{R_Y}(A), cl_{R_Y}(B)$ are closed in Y then by remark 1.12, $cl_R(A)$ and $cl_R(B)$ are closed in X implies

$cl_R(A) \cap Y, cl_R(B) \cap Y$ are closed in X. Since Y is a closed closure subspace of X, by using theorem 1.14, $cl_{R_Y}(A) = cl_R(A) \cap Y, cl_{R_Y}(B) = cl_R(B) \cap Y$ which implies $cl_{R_Y}(A), cl_{R_Y}(B)$ are closed in X. By strong normality of X, there exist disjoint $\langle x\rangle R$ and $\langle y\rangle R$ in X such that $cl_{R_Y}(A) \subseteq \langle x\rangle R$ and $cl_{R_Y}(B) \subseteq \langle y\rangle R$. Thus $((\langle x\rangle R) \cap Y$ and $((\langle y\rangle R) \cap Y$ are two disjoint open sets in Y containing A and B respectively. Hence Y is normal. □

Definition 2.12. Let R be any binary relation then a closure space (X, cl_R) is said to be regular if for any closed set A and a point $x \notin cl_R(A)$ there exist disjoint open sets U and V containing A and a point x.

Definition 2.13. Let R be any binary relation then a closure space (X, cl_R) is said to be strongly regular if for any closed set A and a point $x \notin A$ there exist distinct u and v such that $x \in \langle u\rangle R$ and $cl_R(A) \subset \langle v\rangle R$.

Observation 2.14. *Every strongly regular space is regular.*

The example below establishes that, converse of the above implication is not true.

Example 2.15. *Let* $X = \{a, b, c, d\}$ *and* R *be any binary relation on* X*, where* $R = \Big\{(a, a),\ (a, b),\ (b, c),\ (c, c),\ (d, d)\Big\}$. *Then* $\langle a\rangle R = \{a, b\}$, $\langle b\rangle R = \{a, b\}$, $\langle c\rangle R = \{c\}$, $\langle d\rangle R = \{d\}$ *and* $cl_R(\{a\}) = \{a, b\}$, $cl_R(\{b\}) = \{a, b\}$, $cl_R(\{c\}) = \{c\}$, $cl_R(\{d\}) = \{d\}$, $cl_R(\{a, b\}) = \{a, b\}$, $cl_R(\{a, c\}) = \{a, b, c\}$, $cl_R(\{a, d\}) = \{a, b, d\}$, $cl_R(\{b, c\}) = \{a, b, c\}$, $cl_R(\{b, d\}) = \{a, b, d\}$, $cl_R(\{c, d\}) = \{c, d\}$, $cl_R(\{a, b, c\}) = \{a, b, c\}$, $cl_R(\{a,\ b, d\}) = \{a, b, d\}$, $cl_R(\{a, c, d\}) = X$, $cl_R(\{b, c, d\}) = X$, $cl_R(X) = X$, $cl_R(\phi) = \phi$. *Here, the closure space* (X, cl_R) *is regular because for every closed set and a point not belonging to the closed set there exist disjoint open sets separating the point and the closed set but* (X, cl_R) *is not strongly regular because for closed set* $A = \{c, d\} = cl_R(A)$ *and the point '* b *'* d *there doesn't exist* $\langle x\rangle R$ *and* $\langle y\rangle R$ *respectively.*

Remark 2.16. The above Example 2.5 is not regular because for closed set $A = \{b\} = cl_R(A)$ and a point 'a' disjoint from the closed set there doesn't exist disjoint disjoint open sets separating A and "a" respectively.

Example 2.17. *Let* $X = \{a, b, c\}$ *and* R *be any binary relation on* X *where* $R = \Big\{(a, a),\ (a, b),\ (b, a),\ (b, b),\ (c, c)\Big\}$. *Then* $\langle a\rangle R = \{a, b\}$, $\langle b\rangle R = \{a, b\}$, $\langle c\rangle R = \{c\}$. *Here,* (X, cl_R) *is strongly regular because for every closed set and the point not belonging to the closed set there exist disjoint* $\langle x\rangle R$ *and* $\langle y\rangle R$ *containing closed set and the point respectively.*

Remark 2.18. The Example 2.3 discussed above is strongly normal but not strongly regular as for the closed set $A = \{1, 2, 3\}$ and the point "4" there doesn't exist disjoint x and y satisfying the conditions of strong regularity.

Theorem 2.19. *If* (X, cl_R) *is strongly regular then for every point* x *with* $\langle x\rangle R \neq \phi$*, there exists* $\langle y\rangle R$ *such that* $x \in \langle y\rangle R \subset cl_R(\langle y\rangle R) \subset \langle x\rangle R$.

Proof. Let (X, cl_R) be a strongly regular space and let x be a point with $\langle x\rangle R \neq \phi$. Since $\langle x\rangle R$ is open, $A = cl_R(A) = (X - \langle x\rangle R)$ is a closed set and $x \notin cl_R(A)$. Since X is strongly regular, there exist disjoint $\langle y\rangle R$ and $\langle w\rangle R$ such that $x \in \langle y\rangle R$, $A \subset \langle w\rangle R$. Therefore $x \in \langle y\rangle R \subset (X - \langle w\rangle R) \subset (X - A)$ implies $x \in \langle y\rangle R \subset (X -$

$\langle w\rangle R) \subset \langle x\rangle R$. Since $(X-\langle w\rangle R)$ is a closed set containing $\langle y\rangle R$ and $cl_R(\langle y\rangle R)$ is the smallest closed set containing $\langle y\rangle R$, $x \in \langle y\rangle R \subset cl_R(\langle y\rangle R) \subset (X-\langle w\rangle R) \subset (\langle x\rangle R)$ implies $x \in \langle y\rangle R \subset cl_R(\langle y\rangle R) \subset (\langle x\rangle R)$. □

Theorem 2.20. *Every strongly normal T_1-space is strongly regular.*

Proof. Let $A = cl_R(A)$ be a closed set in the closure space (X, cl_R) and $x \notin A$. Here, (X, cl_R) is T_1 and singletons are closed in a T_1-space. So $\{x\}$ and $A = cl_R(A)$ are two disjoint closed subsets of X. By strong normality, there exists disjoint $\langle x\rangle R$ and $\langle y\rangle R$ such that $\{x\} \subset \langle x\rangle R$, $A = cl_R(A) \subset \langle y\rangle R$. Hence X is strongly regular. □

Definition 2.21. [1] Let (X_1, cl_{R_1}) and (X_2, cl_{R_2}) be two closure spaces. A function $f : X_1 \to X_2$ is continuous at $x \in X_1$ if and only if $f(\langle x\rangle R_1) \subseteq \langle f(x)\rangle R_2$. A function from a closure space (X_1, cl_{R_1}) into a closure space (X_2, cl_{R_2}) is said to be continuous on X_1 if and only if it is continuous at each point of X_1.

Theorem 2.22. [1] *Let f be a function from a closure space (X_1, cl_{R_1}) into a closure space (X_2, cl_{R_2}) then the following conditions are equivalent:*

(1) *f is continuous.*
(2) *for every subset A of X_1, $f(cl_{R_1}(A)) \subseteq cl_{R_2}(f(A))$.*
(3) *the inverse image of every closed subset of X_2 is a closed subset of X_1.*
(4) *the inverse image of every open subset of X_2 is an open subset of X_1.*

Definition 2.23. [1] A function $f : (X_1, cl_{R_1}) \to (X_2, cl_{R_2})$ is called open (closed) if the image of an open (closed) subset of X_1 is an open (closed) subset of X_2.

The following Example shows that continuous image of a normal space need not be normal.

Example 2.24. *Let $X_1 = \{a, b, c, d\}$ and $X_2 = \{1, 2, 3, 4\}$ be two sets and $R_1 = \Big\{(a, a),\ (a, b),\ (a, c)\ (b, b), (b, d)\ (b, a),\ (c, d),\ (c, a)\ (d, c),\ (c, a),\ (d, d)\Big\}$ and $R_2 = \Big\{(1, 1),\ (1, 3),\ (2, 2),\ (2, 3),\ (3, 4),\ (4, 4)\Big\}$ be two binary relations on X_1 and X_2 respectively. By Example 2.5 (X_1, cl_{R_1}) is a closure space. Now $\langle 1\rangle R_2 = \{1, 3\}$, $\langle 2\rangle R_2 = \{2, 3\}$, $\langle 3\rangle R_2 = \{3\}$, $\langle 4\rangle R_2 = \{4\}$ then $cl_{R_2}(\{1\}) = \{1\}$, $cl_{R_2}(\{2\}) = \{2\}$, $cl_{R_2}(\{3\}) = \{1, 2, 3\}$, $cl_{R_2}(\{4\}) = \{4\}$, $cl_{R_2}(\{1, 2\}) = \{1, 2\}$, $cl_{R_2}(\{1, 3\}) = \{1, 2, 3\}$, $cl_{R_2}(\{1, 4\}) = \{1, 4\}$, $cl_{R_2}(\{2, 3\}) = \{1, 2, 3\}$, $cl_{R_2}(\{2, 4\}) = \{2, 4\}$, $cl_{R_2}(\{3, 4\}) = X$, $cl_{R_2}(\{1, 2, 3\}) = \{1, 2, 3\}$, $cl_{R_2}(\{1, 2, 4\}) = \{1, 2, 4\}$, $cl_{R_2}(\{1, 3, 4\}) = X$, $cl_{R_2}(\{2, 3, 4\}) = X$, $cl_{R_2}(X) = X$, $cl_{R_2}(\phi) = \phi$. Note that (X_2, cl_{R_2}) is a closure space. The function $f : (X_1, cl_{R_1}) \to (X_2, cl_{R_2})$ defined below is continuous.*

$$f(x) = \begin{cases} 2 & \text{if } x = b, \\ 3 & \text{if } x = a, \\ 1 & \text{if } x = c, \\ 4 & \text{if } x = d. \end{cases}$$

Here, the closure space (X_1, cl_{R_1}) is normal by example 2.5 but (X_2, cl_{R_2}) is not normal because for disjoint closed sets $A = \{1\} = cl_{R_2}(A)$ and $B = \{2\} = cl_{R_2}(B)$ there doesn't exist disjoint open sets separating A and B.

The following Example shows that continuous image of a strongly normal space need not be strongly normal.

Example 2.25. *Let $X_1 = \{a, b, c\}$ and $X_2 = \{1, 2, 3\}$ be two sets where, $R_1 = \Big\{(a, a),\ (b, b)\ (b, c)\ (c, a)\ (c, c)\Big\}$ and $R_2 = \Big\{(1, 1),\ (2, 1),\ (2, 2),\ (3, 1),\ (3, 3)\Big\}$ be two binary relations on X_1 and X_2 respectively. Then $\langle a\rangle R_1 = \{a\}$, $\langle b\rangle R_1 = \{b, c\}$ and $\langle c\rangle R_1 = \{c\}$. Thus, $cl_{R_1}(a) = \{a\}$, $cl_{R_1}(b) = \{b\}$, $cl_{R_1}(c) = \{b, c\}$, $cl_{R_1}(a, b) = \{a, b\}$, $cl_{R_1}(a, c) = X$, $cl_{R_1}(b, c) = \{b, c\}$, $cl_{R_1}(X) = X$, $cl_{R_1}(\phi) = \phi$. Here, (X_1, cl_{R_1}) is a closure space. Now $\langle 1\rangle R_2 = \{1\}$ $\langle 2\rangle R_2 = \{1, 2\}$ $\langle 3\rangle R_2 = \{1, 3\}$. Then $cl_{R_2}(1) = X$ $cl_{R_2}(2) = \{2\}$ $cl_{R_2}(3) = \{3\}$ $cl_{R_2}(1, 2) = X$ $cl_{R_2}(1, 3) = X$ $cl_{R_2}(2, 3) = \{2, 3\}$ $cl_{R_2}(X) = X$ $cl_{R_2}(\phi) = \phi$. Note that, (X_2, cl_{R_2}) is a closure space. The function $f : (X_1, cl_{R_1}) \to (X_2, cl_{R_2})$ defined below is continuous.*

$$f(x) = \begin{cases} 3 & \textit{if } x = b, \\ 2 & \textit{if } x = a, \\ 1 & \textit{if } x = c. \end{cases}$$

Here, the closure space (X_1, cl_{R_1}) is strongly normal but (X_2, cl_{R_2}) is not strongly normal because for disjoint closed sets $A = \{2\} = cl_{R_2}(A)$ and $B = \{3\} = cl_{R_2}(B)$ there doesn't exist disjoint $\langle x\rangle R$ and $\langle y\rangle R$ separating A and B respectively.

Since continuous image of a strongly normal space is not not strongly normal, the following theorem establishes preservation of strong normality under closed continuous onto mapping.

Theorem 2.26. *Let $f : (X_1, cl_{R1}) \to (X_2, cl_{R_2})$ be a continuous closed, surjection and (X_1, cl_{R_1}) is strongly normal then (X_2, cl_{R_2}) is also strongly normal.*

Proof. Let $A = cl_{R_2}(A)$ and $B = cl_{R_2}(B)$ be two disjoint closed sets in X_2. Since f is continuous, $f^{-1}(cl_R(A))$ and $f^{-1}(cl_R(B))$ are disjoint closed sets in X_1. Thus, by strongly normality of X_1, there exist disjoint open sets $\langle p\rangle R_1$ and $\langle q\rangle R_1$ such that $f^{-1}(A) \subset \langle p\rangle R_1$ and $f^{-1}(B) \subset \langle q\rangle R_1$. Since f is a closed map and $(X_1 - \langle p\rangle R_1), (X_1 - \langle q\rangle R_1)$ are closed in X_1 implies $(f(X_1 - \langle p\rangle R_1)$ and $(f(X_1 - \langle q\rangle R_1)$ are closed in X_2. So $X_2 - f(X_1 - \langle p\rangle R_1)$ and $X_2 - f(X_1 - \langle q\rangle R_1)$ are open in X_2. Thus,

$$\begin{aligned} f^{-1}(A) \subset \langle p\rangle R_1 &\Longrightarrow (X_1 - \langle p\rangle R_1) \subset (X_1 - f^{-1}(A)) \\ &\Longrightarrow f(X_1 - \langle p\rangle R_1) \subset f(X_1 - f^{-1}(A)) \\ &\qquad = (X_2 - A) \end{aligned}$$

Thus, $A \subset X_2 - f(X_1 - \langle p\rangle R_1) = \langle x\rangle R_2$. Now $f^{-1}(\langle x\rangle R_2) = f^{-1}(X_2 - f(X_1 - \langle p\rangle R_1) = \langle p\rangle R_1$. Thus there exists an open set $\langle x\rangle R_2$ containing A such that $f^{-1}(\langle x\rangle R_2) \subset \langle p\rangle R_1$. Similarly there exists an open set $\langle y\rangle R_2$ containing B such that $f^{-1}(\langle y\rangle R_2) \subset \langle q\rangle R_1$. Now $f^{-1}(\langle x\rangle R_2) \cap f^{-1}(\langle y\rangle R_2) \subset \langle p\rangle R_1 \cap \langle q\rangle R_1 = \phi$ implies $\langle x\rangle R_2 \cap \langle y\rangle R_2 = \phi$. Hence (X_2, cl_{R_2}) is strongly normal. □

The following Example shows that continuous image of a regular space need not be regular.

Example 2.27. *Let $X_1 = \{a, b, c, d\}$ and $X_2 = \{1, 2, 3, 4\}$ be two sets and $R_1 = \Big\{(a, a),\ (a, b),\ (b, c),\ (c, c),\ (d, d)\Big\}$ and $R_2 = \Big\{(1, 1),\ (1, 2),\ (2, 3),\ (2, 4),\ (3, 4),\ (3, 3),\ (4, 4)\Big\}$ be two binary relations on X_1 and X_2 respectively. By example 2.15 (X_1, cl_{R_1}) is a closure space. Now, $\langle 1\rangle R_2 = \{1, 2\}$, $\langle 2\rangle R_2 = \{1, 2\}$, $\langle 3\rangle R_2 = \{3, 4\}$, $\langle 4\rangle R_2 = \{4\}$ then $cl_{R_2}(1) = \{1, 2\}$, $cl_{R_2}(\{2\}) = \{1, 2\}$, $cl_{R_2}(\{3\}) = \{3\}$,*

$cl_{R_2}(\{4\}) = \{3,4\}$, $cl_{R_2}(\{1,2\}) = \{1,2\}$, $cl_{R_2}(\{1,3\}) = \{1,2,3\}$, $cl_{R_2}(\{1,4\}) = X$, $cl_{R_2}(\{2,3\}) = \{1,2,3\}$, $cl_{R_2}(\{2,4\}) = X$, $cl_{R_2}(\{3,4\}) = \{3,4\}$, $cl_{R_2}(\{1,2,3\}) = \{1,2,3\}$, $cl_{R_2}(\{1,2,4\}) = X$, $cl_{R_2}(\{1,3,4\}) = X$, $cl_{R_2}(\{2,3,4\}) = X$, $cl_{R_2}(X) = X$, $cl_{R_2}(\phi) = \phi$. *The function* $f : (X_1, cl_{R_1}) \to (X_2, cl_{R_2})$ *defined as follows:*

$$f(x) = \begin{cases} 2 & \text{if } x = b, \\ 1 & \text{if } x = a, \\ 3 & \text{if } x = c, \\ 4 & \text{if } x = d. \end{cases}$$

is continuous.
Here, the closure space (X_1, cl_{R_1}) *is regular but* (X_2, cl_{R_2}) *is not regular because for a closed set* $A = \{3\} = cl_{R_2}(A)$ *and a point* 4 *disjoint from the closed set there doesn't exist disjoint open sets separating* A *and* 4.

The following Example shows that continuous image of a strongly regular space need not be strongly regular.

Example 2.28. *Let* $X_1 = \{a, b, c\}$ *and* $X_2 = \{1, 2, 3\}$ *be two sets where,* $R_1 = \Big\{(a,a),\ (a,b)\ (b,a)\ (b,b)\ (c,c)\Big\}$ *and* $R_2 = \Big\{(1,1),\ (1,3),\ (1,2)\ (2,2),\ (3,1),\ (3,2)\ (3,3)\Big\}$ *be two binary relations on* X_1 *and* X_2 *respectively. Here closure space* (X_1, cl_{R_1}) *is strongly regular by Example 2.17. Now* $\langle 1\rangle R_2 = \{1,2,3\}$ $\langle 2\rangle R_2 = \{2\}$ $\langle 3\rangle R_2 = \{1,2,3\}$. *Then* $cl_{R_2}(1) = \{1,3\}$ $cl_{R_2}(2) = X$ $cl_{R_2}(3) = \{1,3\}$ $cl_{R_2}(1,2) = X$ $cl_{R_2}(1,3) = \{1,3\}$ $cl_{R_2}(2,3) = X$ $cl_{R_2}(X) = X$ $cl_{R_2}(\phi) = \phi$. *Note that* (X_2, cl_{R_2}) *is a closure space. The function* $f : (X_1, cl_{R_1}) \to (X_2, cl_{R_2})$ *defined below is continuous.*

$$f(x) = \begin{cases} 3 & \text{if } x = b, \\ 1 & \text{if } x = a, \\ 2 & \text{if } x = c. \end{cases}$$

Here, (X_1, cl_{R_1}) *is strongly regular but* (X_2, cl_{R_2}) *is not strongly regular because for closed set* $A = \{1,3\} = cl_{R_2}(A)$ *and the point "2" there doesn't exist disjoint* $\langle x\rangle R$ *and* $\langle y\rangle R$ *separating* A *and the point "2" respectively.*

Definition 2.29. [1] A function $f : (X_1, cl_{R_1}) \to (X_2, cl_{R_2})$ is said to be a homeomorphism if f is one-to-one correspondence, continuous and open.

Theorem 2.30. *Let* $f : (X_1, cl_{R1}) \to (X_2, cl_{R_2})$ *be a homeomorphism and* (X_1, cl_{R_1}) *is strongly regular then* (X_2, cl_{R_2}) *is strongly regular.*

Acknowledgment. The first author is thankful to Department of Science and Technology, India for awarding INSPIRE fellowship (IF140967).

References

[1] A. A. Allam, M. Y. Bakeir, E. A. Abo-Tabl, *New approach for closure spaces by relations*, Acta Mathematica Academiae Paedagogicae Nyregyhziensis, 22 (2006), 285-304.
[2] A. A. Allam, M. Y. Bakeir, E. A. Abo-Tabl, *Some methods for generating topologies by relations*, Bulletin of the Malaysian Mathematical Sciences Society, 31 (2008), 1-11
[3] E. Cech, *Topological Spaces*, Academia, (1966).
[4] A. Galton, *A generalized topological view of motion in discrete space*, Theoretical Computer Science, 305 (2003), 111-134.
[5] E. D. Khalimsky, R. Kopperman, P. R. Meyer, *Computer graphics and connected topologies on finite ordered sets*, Topology Appl. 36 (1970), 1-17.

[6] T. Y. Kong, R. Kopperman, P. R. Meyer, *A topological approach to digital topology*, Amer. Math. Monthly, 98 (1991), 902-917.
[7] Guilong Liu, *Closures and topological closures in quasi-discrete closure spaces*, Applied Mathematics Letters, 23 (2010), 772-776.
[8] A. Rosenfeld, *connectivity in digital pictures*, (1970), 146-160.
[9] A. Rosenfeld, *Digital topology*, (1979), 621-630.
[10] J. Slapal, *Closure operations for digital topology*, Theoretical Computer Science, 305 (2003), 457-471.
[11] M. B. Smyth, *Semi-metrics, closure spaces and digital topology*, Theoretical computer science, 151 (1995), 257-276.

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